2024 Maths T 954 ~~S2~~ P2. (Section A).

① $f(x) = \begin{cases} \dfrac{64-x^3}{x-4}, & x<4, \\ (1-m^2)x^2, & x=4, \\ n\sqrt{x-1}, & x>4. \end{cases}$

Give $f(x)$ is continuous at $x=4$; $m, n = ?$

$f(4) = (1-m^2)(4^2)$

$= 16(1-m^2)$.

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$ as $f(x)$ is continuous at $x=4$, $\lim_{x\to 4} f(x)$ exists.

$\lim_{x\to 4^-} \dfrac{64-x^3}{x-4} = \lim_{x\to 4^+} n\sqrt{x-1}$; ~~$64-x^3=($~~

$$\begin{array}{r} -x^2-4x-16 \\ x-4\,\overline{)\,-x^3+64} \\ -(-x^3+4x^2) \\ \hline -4x^2+64 \\ -(-4x^2+16x) \\ \hline -16x+64 \\ -(-16x+64) \\ \hline 0 \end{array}$$

$\lim_{x\to 4^-} (-x^2-4x-16) = n\sqrt{3}$.

$-4^2-4(4)-16 = n\sqrt{3}$

$n = -16\sqrt{3}$ //.

As $f(x)$ is continuous at $x=4$,

$\lim_{x\to 4^-} \dfrac{64-x^3}{x-4} = f(4)$

$-4^2-4(4)-16 = 16(1-m^2)$

$1-m^2 = -3$

$m^2 = 4$

$m=2$ or $m=-2$ //.

② . $x = \dfrac{9t^2-1}{3t}$; $y = \dfrac{9t^2+9t+1}{3t}$; $t \neq 0$.

ⓐ. $x = 3t - \dfrac{1}{3t}$

$$\frac{dx}{dt} = 3 + \frac{1}{3t^2} = \frac{9t^2+1}{3t^2}.$$

$$y = 3t + 3 + \frac{1}{3t}.$$

$$\frac{dy}{dt} = 3 - \frac{1}{3t^2} = \frac{9t^2-1}{3t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{9t^2-1}{3t^2}}{\frac{9t^2+1}{3t^2}}$$

$$= \frac{9t^2-1}{9t^2+1}$$

When the tangent line to the curve is parallel to x-axis,

$$\frac{dy}{dx} = 0$$

$$\frac{9t^2-1}{9t^2+1} = 0 \rightarrow 9t^2 - 1 = 0$$

$$t^2 = \frac{1}{9}$$

$$t = \frac{1}{3} \text{ or } t = -\frac{1}{3}.$$

when $t = -\frac{1}{3}$, $x = \dfrac{9(-\frac{1}{3})^2-1}{3(-\frac{1}{3})} = \dfrac{0}{-1} = 0$

$y = \dfrac{9(-\frac{1}{3})^2+9(-\frac{1}{3})+1}{3(-\frac{1}{3})} = \dfrac{+1-3+1}{-1} = 1 \rightarrow (0, 1).$ ~~(2, 3)~~

when $t = \frac{1}{3}$, $x = \dfrac{9(\frac{1}{3})^2-1}{3(\frac{1}{3})} = 1 - 1 = 0.$

$y = \dfrac{9(\frac{1}{3})^2+9(\frac{1}{3})+1}{3(\frac{1}{3})} = 1 + 3 + 1 = 5 \rightarrow (0, 5)$

Thus, the points are $(0, 5)$ and $(0, 1)$ ~~(2, 3)~~

ⓑ. since $\dfrac{dy}{dx} = \dfrac{9t^2-1}{9t^2+1}$, gradient of tangent to the curve $= \dfrac{dy}{dx}$, ~~$\frac{dy}{dx} \rightarrow \infty$ as $9t^2+1$~~

and $9t^2+1 \neq 0$ as $9t^2 \geq 0$ for all real values of t,

thus $\dfrac{dy}{dx}$ will be defined for all real values of t.

Therefore, there is no tangent line to the curve which is parallel to the y-axis.

(3). $\int_0^{\ln\frac{\pi}{4}} e^x \cot(2e^x)\, dx.$

$= \frac{1}{2}\int_0^{\ln\frac{\pi}{4}} \frac{2e^x\cos(2e^x)}{\sin(2e^x)}\, dx.$

$= \frac{1}{2}\left[\ln|\sin(2e^x)|\right]_0^{\ln\frac{\pi}{4}}$

$= \frac{1}{2}\left(\ln|\sin(2e^{\ln\frac{\pi}{4}})| - \ln|\sin(2e^0)|\right)$

$= \frac{1}{2}\left(\ln|\sin(2\times\frac{\pi}{4})| - \ln|\sin 2|\right)$

$= \frac{1}{2}\left(\ln|\sin\frac{\pi}{2}| - \ln|\sin 2|\right)$

$= \frac{1}{2}\ln\left|\frac{\sin\frac{\pi}{2}}{\sin 2}\right|$

$= \frac{1}{2}\ln\left(\frac{1}{\sin 2}\right) = \frac{1}{2}\ln[\operatorname{cosec} 2]$ //.

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(4). $x^2 e^{x^4}\frac{dy}{dx} + 5xye^{x^4} = 1.$ —— (1)

(1) $\div x^2e^{x^4}$: $\frac{dy}{dx} + \left(\frac{5}{x}\right)y = \frac{1}{x^2e^{x^4}}$: linear differential equation —— (2)

$p(x) = \frac{5}{x}.$

integrating factor, $F = e^{\int\frac{5}{x}dx}$

$= e^{5\ln x}$; given $x > 0$.

$= x^5.$

(2) $\times F$:

$x^5\frac{dy}{dx} + 5x^4\cdot y = x^3e^{-x^4}.$

$\frac{d}{dx}x^5y = x^3e^{-x^4}$

$x^5y = \int x^3e^{-x^4}dx.$

$x^5y = -\frac{1}{4}\int -4x^3e^{-x^4}dx$

$x^5y = -\frac{1}{4}e^{-x^4} + c$

given when $x=1, y=0$: $0 = -\frac{1}{4}e^{-1} + c \rightarrow c = \frac{1}{4e}.$

Thus, $y = \frac{1}{4ex^5} - \frac{1}{4x^5e^{x^4}}$ (particular solution).

(5).

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \dots$$

$$e^x(1+\cos 2x)$$

$$= \left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots\right)\left(1 + 1 - \tfrac{1}{2}(2x)^2 + \tfrac{1}{24}(2x)^4 - \dots\right)$$

$$= \left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots\right)\left(2 - 2x^2 + \tfrac{2}{3}x^4 - \dots\right)$$

$$= 2 - 2x^2 + \tfrac{2}{3}x^4 + 2x - 2x^3 + x^2 - x^4 + \tfrac{1}{3}x^3 + \tfrac{1}{12}x^4 + \dots$$

$$= 2 + 2x - x^2 - \tfrac{5}{3}x^3 - \tfrac{1}{4}x^4 + \dots$$

$$\lim_{x\to 0} \frac{e^x\cos^2 x - 1}{x} = \lim_{x\to 0} \frac{e^x\left(\frac{1+\cos 2x}{2}\right) - 1}{x}$$

$$= \lim_{x\to 0} \frac{e^x(1+\cos 2x) - 2}{2x}$$

$$= \lim_{x\to 0} \frac{2 + 2x - x^2 - \frac{5}{3}x^3 - \frac{1}{4}x^4 + \dots - 2}{2x}$$

$$= \lim_{x\to 0} \frac{2x - x^2 - \frac{5}{3}x^3 + \dots}{2x}$$

$$= \lim_{x\to 0} \left(1 - \tfrac{1}{2}x - \tfrac{5}{6}x^2 + \dots\right)$$

$$= 1 - 0 - 0$$

$$= 1$$

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(6).

For equation $4e^{-2x} + \sqrt{x+1} = 2.1$ —(1)

$$\sqrt{x+1} = 2.1 - 4e^{-2x}$$

$$x + 1 = (2.1 - 4e^{-2x})^2$$

$$x = (2.1 - 4e^{-2x})^2 - 1$$

$$x_{n+1} = (2.1 - 4e^{-2x_n})^2 - 1$$

Let $x_{n+1} = F(x_n)$

$$= (2.1 - 4e^{-2x_n})^2 - 1 \quad \text{—(2)}$$

$$F'(x_n) = 2(2.1 - 4e^{-2x_n})(-4(-2)e^{-2x_n})$$

$$= 16e^{-2x_n}(2.1 - 4e^{-2x_n})$$

Take the approximate root of the equation (1) as $\frac{3+4}{2} = 3.5$

Since $|F'(3.5)| = |16e^{-2(3.5)}(2.1 - 4e^{-2(3.5)})| = 0.0306 < 1$, thus the iteration (2) converges on [3, 4] (cont.).

⑥. (cont.)

Let $x_0 = 4.0$

$x_1 = (2.1 - 4e^{-2(4.0)})^2 - 1 \doteqdot 3.40437$

$x_2 = (2.1 - 4e^{-2(3.40437)})^2 - 1 \doteqdot 3.39147 = 3.391$ (3d.p.)

$x_3 = (2.1 - 4e^{-2(3.39147)})^2 - 1 \doteqdot 3.39099 = 3.391$ (3d.p.)

Thus, the root is $x = 3.391$ (3d.p.)//.

Section B

⑦. $Q(1.099, 1.144)$

ⓐ. for $y = (x - \frac{1}{2})^2 + \frac{\pi}{4}$

minimum point $= (\frac{1}{2}, \frac{\pi}{4})$

Thus, $P = (\frac{1}{2}, \frac{\pi}{4})$//.

ⓑ. Area of the bounded region

$$= \int_{0.5}^{1.099} [\tan^{-1}2x - ((x-\tfrac{1}{2})^2 + \tfrac{\pi}{4})]\, dx.$$

$$= \int_{0.5}^{1.099} (1)(\tan^{-1}2x)\, dx - \int_{0.5}^{1.099} [(x-\tfrac{1}{2})^2 + \tfrac{\pi}{4}]\, dx.$$

$$= [(\tan^{-1}2x)(x)]_{0.5}^{1.099} - \int_{0.5}^{1.099} \left(\frac{2}{1+4x^2}\right)(x)\, dx - \left[\frac{(x-\frac{1}{2})^3}{3(1)} + \frac{\pi}{4}x\right]_{0.5}^{1.099}$$

$$= 1.099\tan^{-1}2(1.099) - 0.5\tan^{-1}1 - \frac{2}{8}\int_{0.5}^{1.099} \frac{8x}{1+4x^2}\, dx - \left[\frac{(1.099-\frac{1}{2})^3}{3} + \frac{\pi}{4}(1.099) - 0 - \frac{\pi}{4}(\tfrac{1}{2})\right]$$

$$= 1.099\tan^{-1}2.198 - \tfrac{1}{2}(\tfrac{\pi}{4}) - \tfrac{1}{4}[\ln|1+4x^2|]_{0.5}^{1.099} - \left(\frac{(0.599)^3}{3} + \frac{1.099}{4}\pi - \frac{\pi}{8}\right)$$

$$= 1.099\tan^{-1}2.198 - \frac{\pi}{8} - \frac{1}{4}(\ln|1+4(1.099)^2| - \ln|1+4(\tfrac{1}{2})^2|) - \left(\frac{(0.599)^3}{3} + \frac{1.099}{4}\pi - \frac{\pi}{8}\right)$$

$\doteqdot 0.59685 - 0.54209$

$= 0.0548$ units^2 (4d.p.)//.

(cont.).

7. (cont.)

c. volume generated, V_y ; $y=\tan^{-1}2x \rightarrow 2x=\tan y$

$x=\frac{1}{2}\tan y$.

$= \pi\int_0^{\frac{\pi}{4}} x^2\,dy$

$= \pi\int_0^{\frac{\pi}{4}} \left(\frac{\tan y}{2}\right)^2 dy$

$= \frac{\pi}{4}\int_0^{\frac{\pi}{4}} (\sec^2 y - 1)\,dy$. ; $\tan^2 y = \sec^2 y - 1$

$= \frac{\pi}{4}\left[\tan y - y\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4}\left(\tan\frac{\pi}{4} - \frac{\pi}{4} - \tan 0 + 0\right)$

$= \frac{\pi}{4}\left(1-\frac{\pi}{4}\right)$ units^3 #.

$= \frac{\pi}{16}(4-\pi)$ units^3 //.

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8. ~~$\frac{dx}{dt} = kx$.~~

a) i. $\frac{dx}{dt} = -kx$, ; $t=0$, $x=x_0$.

$\frac{1}{x}dx = -k\,dt$.

$\int\frac{1}{x}dx = -k\int dt$.

$\ln|x| = -kt + c$.

$|x| = e^{-kt+c}$

$x = \pm e^c \cdot e^{-kt}$

$x = Ae^{-kt}$, $A = \pm e^c =$ constant.

given $t=0$, $x=x_0$

$x_0 = A(e^0) \rightarrow A = x_0$

Thus, the solution is : $x = x_0 e^{-kt}$ //.

(cont.)

8. (cont.)

(a). (ii). half life = 5 hours.

$t = 0,\ x = x_0$

$t = 5,\ x = \frac{1}{2}x_0$.

From $x = x_0 e^{-kt}$.

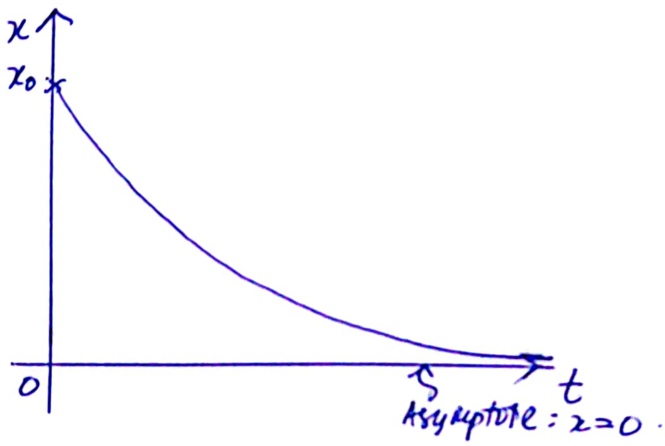
~~when t = 5,~~ $\frac{1}{2}x_0 = x_0 e^{-k(5)}$

$\frac{1}{e^{5k}} = \frac{1}{2}$

$e^{5k} = 2$

$k = \frac{\ln 2}{5}$ //.

(iii). $x = x_0 e^{-\frac{\ln 2}{5}t}$



(b). given $x_0 = 95mg$.

(i). $x = ?$ if $t = 8$ hours.

$x = 95e^{-\frac{\ln 2}{5}(8)}$

$= 31.338$ mg (3 d.p.) //

(ii). $x = x_0 e^{-\frac{\ln 2}{5}t}$

$\frac{x}{x_0} = e^{-\frac{\ln 2}{5}t}$; to eliminate 98% of caffeine from the body,

$\frac{x}{x_0} = \frac{100-98}{100} = \frac{2}{100} = 0.02$.

$0.02 = e^{-\frac{\ln 2}{5}t}$

$\ln 0.02 = -\frac{\ln 2}{5}t$.

$t = \frac{\ln 0.02}{-\frac{\ln 2}{5}} = 28.219$ hours (3d.p.).

Thus, time taken to eliminate 98% of the caffeine from the body is 28.219 hours (3dp) //

The amount of caffeine cannot be totally eliminated from the body as when $t \to \infty$, $x \to 0$ but x will never equal to zero (refer sketch in (a)(iii)) //